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Phase Distribution of Kerr Vectors in a Deformed Hilbert Space

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In this paper we discuss Kerr vectors and their phase distribution in a deformed Hilbert space. We also discuss coherent phase vectors in this space.

1. INTRODUCTION

We consider the set

$$H_q = \{f: f(z) = \Sigma\, a_n z^n \text{ where } \Sigma\, [n]!|a_n|^2 < \infty\}$$

where $[n] = (1 - q^n)/(1 - q)$, $0 < q < 1$.

For $f, g \in H_q$, $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$, $g(z) = \Sigma_{n=0}^{\infty} b_n z^n$ we define addition and scalar multiplication as follows:

$$f(z) + g(z) = \sum_{n=0}^{\infty} (a_n + b_n) z^n \tag{1}$$

and

$$\lambda \circ f(z) = \sum_{n=0}^{\infty} \lambda a_n z^n \tag{2}$$

It is easily seen that H_q forms a vector space with respect to usual pointwise scalar multiplication and pointwise addition by (1) and (2). We observe that $e_q(z) = \Sigma_{n=0}^{\infty} (z^n/[n]!)$ belongs to H_q.

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Now we define the inner product of two functions $f(z) = \Sigma\, a_n z^n$ and $g(z) = \Sigma\, b_n z^n$ belonging to H_q as

$$(f, g) = \Sigma\, [n]!\bar{a}_n b_n \tag{3}$$

The corresponding norm is given by

$$\|f\|^2 = (f, f) = \Sigma\, [n]!|a_n|^2 < \infty$$

With this norm derived from the inner product it can be shown that H_q is a complete normed space. Hence H_q forms a Hilbert space.

In a recent paper[1] we proved that the set $\{z^n / \sqrt{[n]!}, n = 0, 1, 2, 3, \ldots\}$ forms a complete orthonormal set. If we consider the following actions on H_q:

$$\begin{aligned} Tf_n &= \sqrt{[n]}\, f_{n-1} \\ T^*f_n &= \sqrt{[n+1]}\, f_{n+1} \end{aligned} \tag{4}$$

where T is the backward shift and its adjoint T^* is the forward shift operator on H_q and $f_n(z) = z^n / \sqrt{[n]!}$, then we have shown[1] that the solution of the following eigenvalue equation

$$Tf_\alpha = \alpha f_\alpha \tag{5}$$

is given by

$$f_\alpha = e_q\,(|\alpha|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]!}} f_n \tag{6}$$

We call f_α a *coherent vector* in H_q.

This paper is divided into four sections. In Section 1 we have given an introduction stating coherent vectors in H_q. In section 2 we introduce Kerr vectors in H_q. Section 3 deals with phase distributions in H_q and in Section 4 we discuss coherent phase vectors in this space.

2. GENERATION OF KERR VECTORS

The Kerr vectors in H_q are defined by

$$\phi_\alpha^{K} = e_q^{(i/2)\,\gamma N(N-1)} f_\alpha \tag{7}$$

where $f_\alpha \in H_q$ is a coherent vector given by (6), γ is a constant, and $N = T^*T$, with T the backwardshift (4).

Now,

$$\begin{aligned}
\phi_\alpha^{K} &= e_q^{(i/2)\gamma N(N-1)} f_\alpha \\
&= e_q^{(i/2)\gamma N(N-1)} e_q(|\alpha|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]!}} f_n \\
&= e_q(|\alpha|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]!}} e_q^{(i/2)\gamma[n]([n]-1)} f_n \\
&= \sum_{n=0}^{\infty} \left[e_q(|\alpha|^2)^{-1/2} \frac{\alpha^n}{\sqrt{[n]!}} e_q^{(i/2)\gamma[n]([n]-1)} \right] f_n \\
&= \sum_{n=0}^{\infty} q_n f_n
\end{aligned} \tag{8}$$

where

$$q_n = e_q(|\alpha|^2)^{-1/2} \frac{\alpha^n}{\sqrt{[n]!}} e_q^{(i/2)\gamma[n]([n]-1)} \tag{9}$$

The *quasiprobability distribution*, known as the Q function, for Kerr vectors (7) is then defined by

$$\begin{aligned}
Q(\beta) &= |(f_\beta, \phi_\alpha^{K})|^2 \\
&= \left| e_q(|\alpha|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{(\bar{\beta})^n}{\sqrt{[n]!}} q_n \right|^2
\end{aligned} \tag{10}$$

where q_n is given by (9).

3. PHASE DISTRIBUTION

To obtain the phase distribution we consider first the *phase operator* $P = (q^n + T^*T)^{-1/2}T$ and try to find the solution of the following eigenvalue equation:

$$Pf_\beta = \beta f_\beta \tag{11}$$

$$f_\beta(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n \sqrt{[n]!}\, f_n(z) \tag{12}$$

$$\text{or, } f_\beta = \sum_{n=0}^{\infty} a_n \sqrt{[n]!}\, f_n$$

$$Pf_\beta = \sum_{n=0}^{\infty} a_n \sqrt{[n]!}\, (q^n + T^*T)^{-1/2} T f_n$$

$$= \sum_{n=1}^{\infty} a_n \sqrt{[n]!}\,(q^n + T^*T)^{-1/2} \sqrt{[n]}\, f_{n-1}$$

$$= \sum_{n=1}^{\infty} a_n \sqrt{[n]!}\, \sqrt{[n]}\,(q^n + [n-1])^{-1/2} f_{n-1}$$

$$= \sum_{n=0}^{\infty} a_{n+1} \sqrt{[n+1]!}\, \sqrt{[n+1]}\,(q^{n+1} + [n])^{-1/2} f_n \tag{13}$$

$$\beta f_\beta = \beta \sum_{n=0}^{\infty} a_n \sqrt{[n]!}\, f_n \tag{14}$$

From (11)–(14) we observe that a_n satisfies the following difference equation:

$$a_{n+1} \sqrt{[n+1]!}\, \sqrt{[n+1]}\,(q^{n+1} + [n])^{-1/2} = \beta a_n \sqrt{[n]!} \tag{15}$$

That is,

$$a_{n+1} = \frac{\beta a_n (q^{n+1} + [n])^{1/2}}{[n+1]} \tag{16}$$

Hence,

$$a_1 = \frac{\beta (q + [0])^{1/2} a_0}{[1]}$$

$$a_2 = \frac{\beta a_1 (q^2 + [1])^{1/2}}{[2]} = \frac{\beta^2 a_0 \sqrt{(q + [0])(q^2 + [1])}}{[2]!}$$

$$a_3 = \frac{\beta a_2 (q^3 + [2])^{1/2}}{[3]} = \frac{\beta^3 a_0 \sqrt{(q + [0])(q^2 + [1])(q^3 + [2])}}{[3]!}$$

and so on. Thus,

$$a_n = \frac{\beta^n a_0 \sqrt{(q + [0])(q^2 + [1])(q^3 + [2]) \ldots (q^n + [n-1])}}{[n]!}$$

Hence,

$$f_\beta = \sum_{n=0}^{\infty} a_n \sqrt{[n]!} f_n$$

$$= a_0 \sum_{n=0}^{\infty} \beta^n \sqrt{\frac{(q + [0])(q^2 + [1])(q^3 + [2]) \ldots (q^n + [n-1])}{[n]!}}\, f_n$$

where $\beta = |\beta| e^{i\theta}$ is a complex number. These vectors are normalizable in a strict sense only for $|\beta| < 1$.

Now, if we take $a_0 = 1$ and $|\beta| = 1$, we have

$$f_\beta = \sum_{n=0}^{\infty} e^{in\theta} \sqrt{\frac{(q + [0])(q^2 + [1])(q^3 + [2]) \ldots (q^n + [n-1])}{[n]!}} f_n \tag{17}$$

Henceforth, we shall denote this vector as

$$f_\theta = \sum_{n=0}^{\infty} e^{in\theta} \sqrt{\frac{(q + [0])(q^2 + [1])(q^3 + [2]) \ldots (q^n + [n-1])}{[n]!}} f_n \tag{18}$$

$0 \leq \theta \leq 2\pi$, and we call f_θ a *phase vector* in H_q.

The phase vectors f_θ are neither normalizable nor orthogonal. The completeness relation

$$I = \frac{1}{2\pi} \int_0^{2\pi} d\mu(\theta) \, |f_\theta\rangle\langle f_\theta| \tag{19}$$

where

$$d\mu(\theta) = \frac{[n]!}{(q + [0])(q^2 + [1]) \ldots (q^n + [n-1])} d\theta \tag{20}$$

may be proved as follows:

Define the operator

$$|f_\theta\rangle\langle f_\theta|: \quad H_q \rightarrow H_q \tag{21}$$

by

$$|f_\theta\rangle\langle f_\theta| f = (f_\theta, f) f_\theta \tag{22}$$

with

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Now,

$$\begin{aligned} (f_\theta, f) &= \sum_{n=0}^{\infty} [n]! \frac{e^{-in\theta}}{\sqrt{[n]!}} \sqrt{\frac{(q + [0])(q^2 + [1])(q^3 + [2]) \ldots (q^n + [n-1])}{[n]!}} a_n \\ &= \sum_{n=0}^{\infty} e^{-in\theta} \sqrt{(q + [0])(q^2 + [1])(q^3 + [2]) \ldots (q^n + [n-1])} a_n \end{aligned} \tag{23}$$

Then,

$(f_\theta, f) f_\theta$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n \, e^{i(m-n)\theta} \sqrt{\frac{(q + [0])(q^2 + [1]) \ldots (q^m + [m-1])}{[m]!}} \times \sqrt{(q + [0])(q^2 + [1]) \ldots (q^n + [n-1])} \, f_m \tag{24}$$

Using

$$\int_0^{2\pi} d\theta \, e^{i(m-n)\theta} = 2\pi\delta_{mn} \tag{25}$$

we have

$$\begin{aligned}
&\frac{1}{2\pi} \int_0^{2\pi} d\mu(\theta) \, \big|f_\theta\rangle\langle f_\theta\big| f \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n f_m \sqrt{\frac{(q + [0])(q^2 + [1]) \ldots (q^m + [m-1])}{[m]!}} \\
&\quad \times \sqrt{(q + [0])(q^2 + [1]) \ldots (q^n + [n-1])} \, \frac{1}{2\pi} \int_0^{2\pi} e^{i(m-n)\theta} \, d\theta \\
&\quad \times \frac{[n]!}{(q + [0])(q^2 + [1]) \ldots (q^n + [n-1])} \\
&= \sum_{n=0}^{\infty} a_n f_n \frac{(q + [0])(q^2 + [1]) \ldots (q^n + [n-1])}{\sqrt{[n]!}} \\
&\quad \times \frac{[n]!}{(q + [0])(q^2 + [1]) \ldots (q^n + [n-1])} \\
&= \sum_{n=0}^{\infty} \sqrt{[n]!} \, a_n f_n \\
&= f
\end{aligned} \tag{26}$$

Thus, (19) follows.

The *phase distribution* over the window $0 \leq \theta \leq 2\pi$ for any vector f is then defined by

$$P(\theta) = \frac{1}{2\pi} \big|(f_\theta, f)\big|^2 \tag{27}$$

For the Kerr vector, $P(\theta)$ is given by

$$P(\theta) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_n \bar{q}_m \, e^{i(m-n)\theta} \sqrt{(q + [0])(q^2 + [1])(q^3 + [2]) \ldots (q^n + [n-1])} \times \sqrt{(q + [0])(q^2 + [1])(q^3 + [2]) \ldots (q^m + [m-1])} \tag{28}$$

where q_n is given by (9).

4. COHERENT PHASE VECTORS

The coherent phase vectors $\{f_\beta\}$ are the unit-length eigenvectors of the operator P of (11) whose eigenvalues β are complex numbers, namely,

$$Pf_\beta = \beta f_\beta$$

From the previous section we see that

$$f_\beta = \sum_{n=0}^{\infty} a_n \sqrt{[n]!} f_n = a_0 \sum_{n=0}^{\infty} \beta^n \sqrt{\frac{(q + [0])(q^2 + [1])(q^3 + [2]) \ldots (q^n + [n-1])}{[n]!}} f_n \tag{29}$$

Here, $|\beta| < 1$ is necessary in order to ensure normalization, for,

$$\|f_\beta\|^2 = |a_0|^2 \sum_{n=0}^{\infty} |\beta|^{2n} \frac{(q + [0])(q^2 + [1])(q^3 + [2]) \ldots (q^n + [n-1])}{[n]!} \tag{30}$$

The above series (30) is convergent for $|\beta| < 1$. Thus, when f_β is normalized we have $|a_0| = \Phi(|\beta|^2)^{-1/2}$, where

$$\Phi(|\beta|^2) = \sum_{n=0}^{\infty} |\beta|^{2n} \frac{(q + [0])(q^2 + [1])(q^3 + [2]) \ldots (q^n + [n-1])}{[n]!} \tag{31}$$

Thus, f_β (29) takes the form

$$f_\beta = \Phi(|\beta|^2)^{-1/2} \sum_{n=0}^{\infty} \beta^n \sqrt{\frac{(q + [0])(q^2 + [1])(q^3 + [2]) \ldots (q^n + [n - 1])}{[n]!}} f_n \qquad (32)$$

where $|\beta| < 1$.

It is proper to call these vectors *coherent phase vectors* because they are eigenvectors of the *phase operator* P and they are *coherent* with respect to P.

We have that the coherent phase vectors are normalized and nonorthogonal,

$$(f_\alpha, f_\beta) = \frac{\Phi(\bar{\alpha}\beta)}{\sqrt{\Phi(|\alpha|^2)\Phi(|\beta|^2)}} \qquad (33)$$

If we ignore the normalization factor $\Phi(|\beta|^2)^{-1/2}$, then we get

$$f_\beta \to \sum_{n=0}^{\infty} e^{in\theta} \sqrt{\frac{(q + [0])(q^2 + [1])(q^3 + [2]) \ldots (q^n + [n - 1])}{[n]!}} f_n = f_\theta \qquad (34)$$

for $\beta = |\beta| e^{i\theta}$ with $|\beta| \to 1$.

The *phase representation* of the coherent phase vector $f(\beta)$ is

$$(f_\theta, f_\beta) = \Phi(|\beta|^2)^{-1/2} \sum_{n=0}^{\infty} \beta^n e^{-in\theta} \frac{(q + [0])(q^2 + [1])(q^3 + [2]) \ldots (q^n + [n - 1])}{[n]!} = \frac{\Phi(e^{-i\theta}\beta)}{\sqrt{\Phi(|\beta|^2)}} \qquad (35)$$

Now, from (35) we have the *phase distribution* over the window $0 \leq \theta \leq 2\pi$ for the *coherent phase vector* f_β as

$$P(\theta) = \frac{1}{2\pi} |(f_\theta, f_\beta)|^2 = \frac{1}{2\pi} \frac{|\Phi(e^{-i\theta}\beta)|^2}{\Phi(|\beta|^2)} \qquad (36)$$

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